10. Kruskal's algorithm for finding a minimal spanning tree

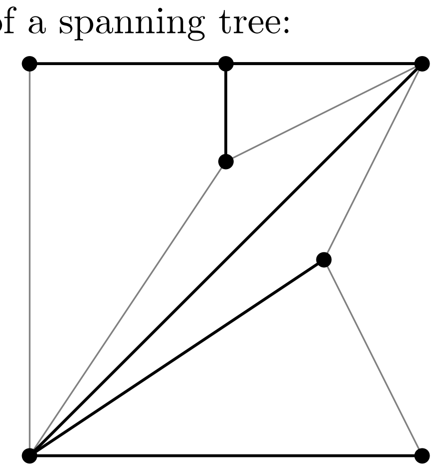
To read:
[1] 9.1. Finding the best tree
[3] 5.4. Minimum spanning tree problem

10.1. Subgraphs, induced subgraphs, and spanning trees.

Definition 10.1. Let G and G' be graphs. We say that G is a subgraph of G' if $V(G) \subset V(G')$ and $E(G) \subseteq E(G')$. We say that G is an induced subgraph of G' if $V(G) \subseteq V(G')$ and $E(G) = E(G') \cap \binom{V(G)}{2}$.

Definition 10.2. Let $G = (V, E)$ be a graph. We say that a tree T is a spanning tree of G if it contains all the vertices of V and is a subgraph of G, that is every edge in the tree belongs to the graph G.

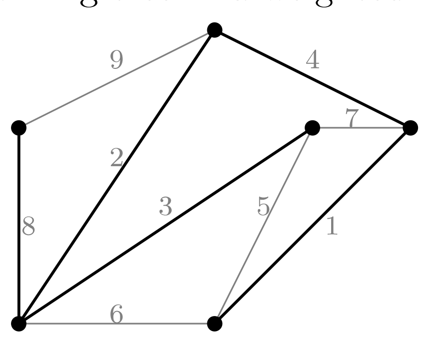
Example. Below is an example of a spanning tree:



10.2. Weighted graphs.

Definition 10.3. A *weighted graph* is a graph in which each edge is given a numerical weight. We define the *weight of a graph* as the sum of the weights of all its edges.

We are interested in the following problem: find a minimum weight spanning tree T for a given weighted connected graph G.
Example. A minimum weight spanning tree in a weighted connected graph.



One way to solve the problem of finding a minimum spanning tree is using Kruskal's algorithm. This works as follows:

Step 1. Start with an empty graph.
Step 2. Take all the edges that have not been selected and that would not create a cycle with the already selected edges and select it unless it creates a cycle. Add the one with the smallest weight.
Step 3. Repeat until the graph is connected.

Theorem 10.4. ***(Correctness of Kruskal's algorithm).*** *The Kruskal's algorithm solves the minimum spanning tree problem.*

Proof. The proof can be found in [1] Chapter 9.1. □

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References

[1] Discrete Mathematics (L. Lovasz, J. Pelikan , K. Vesztergombi);
[2] Combinatorics: Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability (B. Bollobas);
[3] Invitation to Discrete Mathematics (J. Matousek, J. Nesetril).
[4] Extremal combinatorics (S. Jukna).
[5] Combinatorial theory (M. Hall), Blaisdell publishing company, 1967.